

Foundations
of
Game Engine Development

VOLUME 2
RENDERING

Solutions Guide

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Chapter 5

1. The following table shows the storage requirements, in bytes, for 32-bit and 64-bit pixels.

Display Resolution	32-bit Pixels	64-bit Pixels
640×480	1,228,800	2,457,600
1024×768	3,145,728	6,291,456
1280×720	3,686,400	7,372,800
1280×1024	5,242,880	10,485,760
1920×1080	8,294,400	16,588,800
2560×1440	14,745,600	29,491,200
3840×2160	33,177,600	66,355,200
7680×4320	132,710,400	265,420,800

2. A CIE RGB color C_{CIE} is transformed into an sRGB color C_{sRGB} with

$$C_{\text{sRGB}} = \begin{bmatrix} 7.43651 & -1.40848 & 0.781042 \\ -0.807764 & 6.91648 & -0.750184 \\ -0.0499435 & -0.77922 & 5.96362 \end{bmatrix} C_{\text{CIE}}.$$

The reverse transformation is given by

$$C_{\text{CIE}} = \begin{bmatrix} 0.137229 & 0.0262932 & -0.0146651 \\ 0.0163836 & 0.1498 & 0.0166981 \\ 0.00328997 & 0.0197934 & 0.169742 \end{bmatrix} C_{\text{sRGB}}.$$

3. $(x_R, y_R) = (0.735, 0.265)$
 $(x_G, y_G) = (0.274, 0.717)$
 $(x_B, y_B) = (0.167, 0.00886)$
4. The matrix that converts from the XYZ color space to the uRGB color space is given by

$$\mathbf{M}_{\text{uRGB}} = \begin{bmatrix} 1.716651 & -0.355671 & -0.253366 \\ -0.666684 & 1.616481 & 0.015769 \\ 0.017640 & -0.042771 & 0.942103 \end{bmatrix}.$$

When we multiply this by the matrix $\mathbf{M}_{\text{sRGB}}^{-1}$ that converts from sRGB to XYZ, we obtain as the matrix that converts directly from sRGB to uRGB

$$\mathbf{M}_{\text{sRGB} \rightarrow \text{uRGB}} = \begin{bmatrix} 0.627404 & 0.329283 & 0.043313 \\ 0.069097 & 0.919540 & 0.011362 \\ 0.016391 & 0.088013 & 0.895595 \end{bmatrix}.$$

The inverse of this matrix, converting from uRGB to sRGB is

$$\mathbf{M}_{\text{uRGB} \rightarrow \text{sRGB}} = \begin{bmatrix} 1.660491 & -0.587641 & -0.072850 \\ -0.124550 & 1.132900 & -0.008349 \\ -0.018151 & -0.100579 & 1.118730 \end{bmatrix}.$$

5. $g(x) = 0.274882x^3 + 0.725118x^2$
6. The matrix that converts from W to the coordinate system in which x points east and y points up is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},$$

and the inverse converting back to W is

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The matrix that converts from W to the coordinate system in which x points east and z points up is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and this matrix is equal to its inverse.

7. The world-space direction is equal to $\mathbf{W}_{\text{skull}} [1 \ 0 \ 0]^T$, where $\mathbf{W}_{\text{skull}}$ is the matrix given by Equation (5.25). This is simply the first column of $\mathbf{W}_{\text{skull}}$.

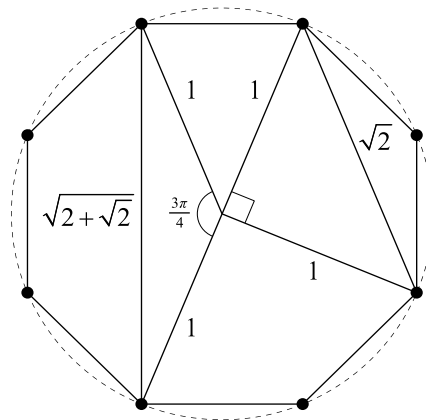
$$8. \quad x = \frac{p_x(p_z - q_z) - p_z(p_x - q_x)}{p_w(p_z - q_z) - p_z(p_w - q_w)} \quad \text{and} \quad y = \frac{p_y(p_z - q_z) - p_z(p_y - q_y)}{p_w(p_z - q_z) - p_z(p_w - q_w)}.$$

9. Let $\mathbf{u} = \mathbf{p}_1 - \mathbf{p}_0$ and $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_0$. Then $\mathbf{u} = u_x \mathbf{e}_1 + u_y \mathbf{e}_2$ and $\mathbf{v} = v_x \mathbf{e}_1 + v_y \mathbf{e}_2$. The wedge product is

$$\mathbf{u} \wedge \mathbf{v} = (u_x v_y - u_y v_x) (\mathbf{e}_1 \wedge \mathbf{e}_2),$$

where the presence of the 2D bivector basis element $\mathbf{e}_1 \wedge \mathbf{e}_2$ corresponds to the fact that we have calculated an area. This is the area of the parallelogram having sides \mathbf{u} and \mathbf{v} . The area of a triangle with sides \mathbf{u} and \mathbf{v} is half of that, which has the magnitude equivalent to Equation (5.35).

10. There are three possible lengths for each interior edge. An edge connecting vertices spaced two apart is the hypotenuse of the right triangle in the figure below, and it has length $\sqrt{2}$. An edge connecting vertices spaced four apart is a diameter of the octagon and thus has length 2. The remaining possibility, an edge connecting vertices spaced three apart is the side opposite the angle $\frac{3\pi}{4}$ in the figure. By the law of cosines, its length is $\sqrt{2 + \sqrt{2}}$. The total length of all five interior edges is minimized when there are four edges having length $\sqrt{2}$ and one edge having length 2.



Chapter 6

1. The following table lists the $[\mathbf{n} | d]$ components of the world-space frustum planes. Note that these are not normalized.

Frustum Plane	World-Space \mathbf{n}	World-Space d
Near	$\mathbf{M}_{\text{camera}[2]}$	$-n - \mathbf{n} \cdot \mathbf{M}_{\text{camera}[3]}$
Far	$-\mathbf{M}_{\text{camera}[2]}$	$f - \mathbf{n} \cdot \mathbf{M}_{\text{camera}[3]}$
Left	$g\mathbf{M}_{\text{camera}[0]} + s\mathbf{M}_{\text{camera}[2]}$	$-\mathbf{n} \cdot \mathbf{M}_{\text{camera}[3]}$
Right	$-g\mathbf{M}_{\text{camera}[0]} + s\mathbf{M}_{\text{camera}[2]}$	$-\mathbf{n} \cdot \mathbf{M}_{\text{camera}[3]}$
Top	$g\mathbf{M}_{\text{camera}[1]} + \mathbf{M}_{\text{camera}[2]}$	$-\mathbf{n} \cdot \mathbf{M}_{\text{camera}[3]}$
Bottom	$-g\mathbf{M}_{\text{camera}[1]} + \mathbf{M}_{\text{camera}[2]}$	$-\mathbf{n} \cdot \mathbf{M}_{\text{camera}[3]}$

2. The Plücker coordinates $\{\mathbf{v} | \mathbf{m}\}$ are given by the following formulas, where the notation $[\mathbf{n}_r | d_r]$ means the \mathbf{n} and d components of the plane \mathbf{f} .

$$\begin{aligned} & \{\mathbf{n}_l \times \mathbf{n}_t | d_l \mathbf{n}_t - d_t \mathbf{n}_l\} \\ & \{\mathbf{n}_t \times \mathbf{n}_r | d_t \mathbf{n}_r - d_r \mathbf{n}_t\} \\ & \{\mathbf{n}_r \times \mathbf{n}_b | d_r \mathbf{n}_b - d_b \mathbf{n}_r\} \\ & \{\mathbf{n}_b \times \mathbf{n}_l | d_b \mathbf{n}_l - d_l \mathbf{n}_b\} \end{aligned}$$

Bivectors corresponding to the same lines are given by $\mathbf{t} \vee \mathbf{l}$, $\mathbf{r} \vee \mathbf{t}$, $\mathbf{b} \vee \mathbf{r}$, and $\mathbf{l} \vee \mathbf{b}$.

3. $\left(\pm \frac{ns}{g}, \pm \frac{n}{g}, n\right)$ and $\left(\pm \frac{fs}{g}, \pm \frac{f}{g}, f\right)$.
4. $r = \frac{h}{g} \sqrt{s^2 + 1}$ and $\sin \alpha = \sqrt{\frac{s^2 + 1}{g^2 + s^2 + 1}}$.
5. n/g

$$6. \mathbf{P}_{\text{infinite}}^{-1} = \begin{bmatrix} s/g & 0 & 0 & 0 \\ 0 & 1/g & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{n} & \frac{1}{n} \end{bmatrix}, (\mathbf{P}_{\text{infinite}}^*)^{-1} = \begin{bmatrix} s/g & 0 & 0 & 0 \\ 0 & 1/g & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{n(1-\varepsilon)} & \frac{1}{n} \end{bmatrix},$$

$$\mathbf{R}_{\text{frustum}}^{-1} = \begin{bmatrix} s/g & 0 & 0 & 0 \\ 0 & 1/g & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{n-f}{nf} & \frac{1}{f} \end{bmatrix}, \mathbf{R}_{\text{infinite}}^{-1} = \begin{bmatrix} s/g & 0 & 0 & 0 \\ 0 & 1/g & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{n} & 0 \end{bmatrix}, \text{ and}$$

$$(\mathbf{R}_{\text{infinite}}^*)^{-1} = \begin{bmatrix} s/g & 0 & 0 & 0 \\ 0 & 1/g & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{n(1-\varepsilon)} & -\frac{\varepsilon}{n(1-\varepsilon)} \end{bmatrix}.$$

$$7. \begin{bmatrix} \frac{2g}{r-l} & 0 & -\frac{r+l}{r-l} & 0 \\ 0 & \frac{2g}{b-t} & -\frac{b+t}{b-t} & 0 \\ 0 & 0 & \frac{f}{f-n} & -\frac{nf}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$8. \begin{bmatrix} g/s & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & \frac{f+n}{f-n} & -\frac{2nf}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$9. \begin{bmatrix} g/s & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 1 & -2n \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

10. The w component of $\mathbf{v}_{\text{camera}}$ is given by

$$\frac{1 - P_{22}}{P_{23}} = \frac{1}{f}$$

because $P_{22} = f/(f - n)$ and $P_{23} = -nf/(f - n)$. The dot product $\mathbf{k} \cdot \mathbf{v}_{\text{camera}}$ only involves the z and w components of $\mathbf{v}_{\text{camera}}$ in this case, and it is given by

$$\mathbf{k} \cdot \mathbf{v}_{\text{camera}} = 1 - \frac{n}{f}.$$

Taking the reciprocal, $m = f/(f - n)$, and $m\mathbf{k}$ thus recovers $\mathbf{P}_{\text{frustum}}$.

11. For any near plane \mathbf{k} , the far plane \mathbf{f} is given by

$$\mathbf{f} = (-k_x, -k_y, 1 - k_z, -k_w).$$

The moment \mathbf{m} of the line where \mathbf{k} and \mathbf{f} intersect is given by $k_w \mathbf{f}_{xyz} - f_w \mathbf{k}_{xyz}$, which is always $(0, 0, k_w)$. This means that the plane containing the line $\{\mathbf{v} \mid \mathbf{m}\}$ and the origin is $(0, 0, k_w, 0)$, which is the x - y plane.

12. The scalar field $z_{\text{device}}(\mathbf{p})$ is given by

$$z_{\text{device}}(\mathbf{p}) = \frac{1}{p_z} (k_x p_x + k_y p_y + k_z p_z + k_w).$$

Its gradient is then

$$\nabla z_{\text{device}}(\mathbf{p}) = \left(\frac{k_x}{p_z}, \frac{k_y}{p_z}, -\frac{k_x p_x + k_y p_y + k_w}{p_z^2} \right),$$

which we abbreviate ∇z . The direction \mathbf{v} of the line where the near plane \mathbf{k} and far plane \mathbf{f} intersect is given by

$$\mathbf{v} = \mathbf{k}_{xyz} \times \mathbf{f}_{xyz} = (k_y, -k_x, 0),$$

and the moment \mathbf{m} of the line is given by $\mathbf{m} = (0, 0, k_w)$ as in the previous exercise. Clearly, $\nabla z \cdot \mathbf{v} = 0$, so the gradient is perpendicular to the direction \mathbf{v} . The point \mathbf{s} on the line $\{\mathbf{v} \mid \mathbf{m}\}$ closest to the origin is given by

$$\mathbf{s} = (\mathbf{v} \times \mathbf{m}) / v^2 = -\frac{k_w}{k_x^2 + k_y^2} (k_x, k_y, 0),$$

and the point \mathbf{q} must be equal to $\mathbf{s} + t\mathbf{v}$ for some t . Since $\nabla z \cdot \mathbf{v} = 0$, the values of $\nabla z \cdot (\mathbf{p} - \mathbf{s})$ and $\nabla z \cdot (\mathbf{p} - \mathbf{q})$ are the same. After plugging in the values of ∇z and \mathbf{s} , the dot product $\nabla z \cdot (\mathbf{p} - \mathbf{s})$ simplifies to zero, so the gradient is perpendicular to $\mathbf{p} - \mathbf{q}$.

13. The near plane is now $(0, 0, 1, 1)$ in clip space, and this corresponds to $\mathbf{P}_3 + \mathbf{P}_2$ in camera space. This means that $\mathbf{P}_2 = m\mathbf{k} - \mathbf{P}_3$ and thus $\mathbf{f} = \mathbf{P}_3 - \mathbf{P}_2 = 2\mathbf{P}_3 - m\mathbf{k}$. When we solve $\mathbf{f} \cdot \mathbf{v}_{\text{camera}} = 0$ for m , we get

$$m = \frac{2}{\mathbf{k} \cdot \mathbf{v}_{\text{camera}}}.$$

Then $\mathbf{P}_2 = (mk_x, mk_y, mk_z - 1, mk_w)$.

14. The modified projection matrix transforms \mathbf{p} as

$$\begin{bmatrix} A & B & C + \delta_{\text{device}} & D \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} Ap_x + Bp_y + (C + \delta_{\text{device}})p_z + D \\ p_z \end{bmatrix}.$$

The original projection matrix transforms $(p_x, p_y, p_z + \delta_{\text{camera}})$ as

$$\begin{bmatrix} A & B & C & D \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z - \delta_{\text{camera}} \\ 1 \end{bmatrix} = \begin{bmatrix} Ap_x + Bp_y + C(p_z - \delta_{\text{camera}}) + D \\ p_z - \delta_{\text{camera}} \end{bmatrix}.$$

Setting the results equal to each other after the perspective divide, we have

$$\frac{Ap_x + Bp_y + D}{p_z} + C + \delta_{\text{device}} = \frac{Ap_x + Bp_y + D}{p_z - \delta_{\text{camera}}} + C.$$

Solving for δ_{device} gives us

$$\delta_{\text{device}} = \frac{(Ap_x + Bp_y + D)\delta_{\text{camera}}}{p_z(p_z - \delta_{\text{camera}})}.$$

Chapter 7

1. The vectors \mathbf{r} and \mathbf{s} are defined as

$$\mathbf{r} = 2(\mathbf{n} \cdot \mathbf{l})\mathbf{n} - \mathbf{l} \quad \text{and} \quad \mathbf{s} = 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n} - \mathbf{v}.$$

Then $\mathbf{r} \cdot \mathbf{v} = 2(\mathbf{n} \cdot \mathbf{l})(\mathbf{n} \cdot \mathbf{v}) - \mathbf{l} \cdot \mathbf{v}$, and $\mathbf{s} \cdot \mathbf{l} = 2(\mathbf{n} \cdot \mathbf{v})(\mathbf{n} \cdot \mathbf{l}) - \mathbf{v} \cdot \mathbf{l}$, which have the same value.

$$\begin{aligned} 2. \quad (\mathbf{n} \cdot \mathbf{h})^\alpha &= \left(\mathbf{n} \cdot \frac{\mathbf{l} + \mathbf{v}}{\|\mathbf{l} + \mathbf{v}\|} \right)^\alpha = \frac{(\mathbf{n} \cdot \mathbf{l} + \mathbf{n} \cdot \mathbf{v})^\alpha}{[(\mathbf{l} + \mathbf{v}) \cdot (\mathbf{l} + \mathbf{v})]^{\alpha/2}} \\ &= \frac{(\mathbf{n} \cdot \mathbf{l} + \mathbf{n} \cdot \mathbf{v})^\alpha}{[l^2 + v^2 + 2(\mathbf{l} \cdot \mathbf{v})]^{\alpha/2}} \\ &= \left[\frac{(\mathbf{n} \cdot \mathbf{l} + \mathbf{n} \cdot \mathbf{v})^2}{2(\mathbf{l} \cdot \mathbf{v} + 1)} \right]^{\alpha/2} \end{aligned}$$

3. In addition to the constraints $\mathbf{s} \cdot \mathbf{e}_1 = x_1$ and $\mathbf{s} \cdot \mathbf{e}_2 = x_2$, we know that $\mathbf{s} \cdot \mathbf{n} = 0$. This gives us the 3×3 system

$$\begin{bmatrix} \leftarrow & \mathbf{e}_1 & \rightarrow \\ \leftarrow & \mathbf{e}_2 & \rightarrow \\ \leftarrow & \mathbf{n} & \rightarrow \end{bmatrix} \mathbf{s} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Using the transpose of Equation (1.95) for the inverse of the matrix, we have

$$\mathbf{s} = \frac{1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}]} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{e}_2 \times \mathbf{n} & \mathbf{n} \times \mathbf{e}_1 & \mathbf{e}_1 \times \mathbf{e}_2 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

From Equation (7.37), we have

$$\begin{aligned} \mathbf{t} &= \frac{1}{x_1 y_2 - x_2 y_1} (y_2 \mathbf{e}_1 - y_1 \mathbf{e}_2) \\ \mathbf{b} &= \frac{1}{x_1 y_2 - x_2 y_1} (x_1 \mathbf{e}_2 - x_2 \mathbf{e}_1). \end{aligned}$$

The dot product $\mathbf{s} \cdot \mathbf{t}$ is then

$$\begin{aligned}\mathbf{s} \cdot \mathbf{t} &= \frac{1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}](x_1 y_2 - x_2 y_1)} ((\mathbf{e}_2 \times \mathbf{n})x_1 + (\mathbf{n} \times \mathbf{e}_1)x_2) \cdot (y_2 \mathbf{e}_1 - y_1 \mathbf{e}_2) \\ &= \frac{1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}](x_1 y_2 - x_2 y_1)} ([\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}]x_1 y_2 - [\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}]x_2 y_1) \\ &= 1,\end{aligned}$$

and the dot product $\mathbf{s} \cdot \mathbf{b}$ is

$$\begin{aligned}\mathbf{s} \cdot \mathbf{b} &= \frac{1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}](x_1 y_2 - x_2 y_1)} ((\mathbf{e}_2 \times \mathbf{n})x_1 + (\mathbf{n} \times \mathbf{e}_1)x_2) \cdot (x_1 \mathbf{e}_2 - x_2 \mathbf{e}_1) \\ &= \frac{1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}](x_1 y_2 - x_2 y_1)} ([\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}]x_2 x_1 - [\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}]x_1 x_2) \\ &= 0.\end{aligned}$$

4. Define $\mathbf{t} = t_x \mathbf{e}_1 + t_y \mathbf{e}_2 + t_z \mathbf{e}_3$ and $\mathbf{n} = n_x \bar{\mathbf{e}}_1 + n_y \bar{\mathbf{e}}_2 + n_z \bar{\mathbf{e}}_3$. Then $\bar{\mathbf{t}}$ is a bivector given by $\bar{\mathbf{t}} = t_x \bar{\mathbf{e}}_1 + t_y \bar{\mathbf{e}}_2 + t_z \bar{\mathbf{e}}_3$. The antiwedge product $\mathbf{n} \vee \bar{\mathbf{t}}$ is

$$\mathbf{n} \vee \bar{\mathbf{t}} = (n_y t_z - n_z t_y) \bar{\mathbf{e}}_{23} + (n_z t_x - n_x t_z) \bar{\mathbf{e}}_{31} + (n_x t_y - n_y t_x) \bar{\mathbf{e}}_{12},$$

which is equivalent to the vector \mathbf{b} given by $\mathbf{n} \times \mathbf{t}$.

5. $f_{\text{scale}}(\mathbf{m}, s) = \text{normalize}(sm_x, sm_y, m_z)$

Chapter 8

1. 2.25×10^{27} cd
2. The plane \mathbf{f} has two nonzero coordinates and can be written $\mathbf{f} = (f_x, 0, f_z, 0)$. From the dot product $\mathbf{f} \cdot \mathbf{l} = r$, we can write $f_z l_z = r - f_x l_x$. Squaring both sides and substituting $f_z^2 = 1 - f_x^2$ (because \mathbf{f} is normalized), we have

$$(1 - f_x^2)l_z^2 = r^2 - 2rf_x l_x + f_x^2 l_x^2.$$

Rewriting as a quadratic equation in f_x gives us

$$(l_x^2 + l_z^2)f_x^2 - 2rl_x f_x + r^2 - l_z^2 = 0.$$

The values of f_x are then given by

$$f_x = \frac{b \pm \sqrt{b^2 - ac}}{a} \quad \text{or} \quad f_x = \frac{c}{b \pm \sqrt{b^2 - ac}},$$

where $a = l_x^2 + l_z^2$, $b = rl_x$, and $c = r^2 - l_z^2$. For each value of f_x , the corresponding value of f_z is given by

$$f_z = \frac{r - f_x l_x}{l_z}.$$

The coordinate q'_x where each plane \mathbf{f} intersects the plane $z = g$ is then

$$q'_x = -g \frac{f_z}{f_x}.$$

To show that this is equal to the value of q_x given by Equation (8.29), we calculate the coordinate p'_x where \mathbf{f} intersects the plane $z = l_z$ and show that it is the same value of p_x from which q_x is derived. The coordinate p'_x is given by

$$p'_x = -l_z \frac{f_z}{f_x} = \frac{f_x l_x - r}{f_x} = l_x - \frac{r}{f_x}.$$

Using this value of p'_x , the distance u used in Equation (8.28) is

$$u = p'_x - l_x = \frac{r}{f_x}.$$

When we substitute the above solutions for f_x having the radical in the denominator, we find that

$$u = \frac{l_x r^2 \pm \sqrt{l_x^2 r^4 + r^2 (l_z^2 - r^2) (l_x^2 + l_z^2)}}{l_z^2 - r^2}.$$

These are exactly the same values of u given by Equation (8.28), so the corresponding values of q_x must be equal to those produced by the values of q'_x given above.

- Plugging $\mathbf{k} = (0, 0, 1, -n)$ into Equation (8.43) gives us $d = n^2 r^2$. The value of a in Equation (8.41) does not change, but we now have $b = (l_z - n)l_z - r^2$. Using these values in Equation (8.42) yields

$$z_{\text{device}} = \frac{l_z^2 - l_z n - r^2 \pm rn}{l_z^2 - r^2}.$$

This can be rewritten as

$$z_{\text{device}} = 1 - \frac{n(l_z \pm r)}{(l_z + r)(l_z - r)}.$$

Choosing different signs in the numerator causes a different factor to be cancelled in the denominator, so this is equivalent to

$$z_{\text{device}} = 1 - \frac{n}{l_z \pm r}.$$

4. Let the notation $[pq]$ represent the distance between vertices \mathbf{v}_p^k and \mathbf{v}_q^k . By symmetry, $[06] = [17] = [24] = [35]$ and $[46] = [57]$. We need to show that the distances between each of the 22 remaining pairs of vertices is less than either $[06]$ or $[46]$. First, the length of each edge on the near plane, such as $[01]$, is shorter than the diagonal length $[02]$, and the length of each edge on the far plane, such as $[45]$, is short than the diagonal length $[46]$. Because the edges on the far plane must be longer, $[02] < [46]$. At this point, we have eliminated the six possibilities on the near plane and the four edges on the far plane. The remaining 12 possibilities lie on the lateral planes. Each of the four lateral edge lengths such as $[04]$ is shorter than the diagonal length of a lateral side such as $[05]$, which has the same length as $[14]$. Each of these eight diagonal lengths is shorter than the interior diagonal length $[06]$. This accounts for all cases.
5. There are four possible cases. First, if $u_1, u_2,$ and u_3 are all negative, then $w = 1$ and $L = L_0$. Second, if $u_1 \geq 0$ but u_2 and u_3 are negative, then $w = 1 - \text{sat}(u_1)$, and

$$L = (1 - \text{sat}(u_1))L_0 + \text{sat}(u_1)L_1,$$

which correctly blends from cascade 0 to cascade 1. Third, if $u_1 \geq 0$ and $u_2 \geq 0$ but u_3 is negative, then $w = \text{sat}(u_2)$, and

$$L = (1 - \text{sat}(u_2))L_1 + \text{sat}(u_2)L_2,$$

which correctly blends from cascade 1 to cascade 2. Finally, if $u_1, u_2,$ and u_3 are all nonnegative, then $w = 1 - \text{sat}(u_3)$ (using the fact that cascade 3 begins beyond the end of cascade 1), and

$$L = (1 - \text{sat}(u_3))L_2 + \text{sat}(u_3)L_3,$$

which correctly blends from cascade 2 to cascade 3.

6. Suppose that all three inequalities are true. Then combining $i < j$ and $j < k$, we must have $i < k$. But it must also be true, by the third inequality, that $k < i$, which is a contradiction. Now suppose that none of the three inequalities is true. Then $i > j$, $j > k$, and $k > i$. Again, we have a contradiction because $i > k$ and $k > i$ would both be required to be true. Thus, it can only be the case the either one or two of the inequalities is true.
7. We need to integrate from the camera position c to the point where the ray $\mathbf{q}(t) = c + t\mathbf{d}$ intersects the plane \mathbf{f} . This gives us

$$\tau(\mathbf{d}) = -\alpha_0 \int_0^{t_0} \mathbf{f} \cdot (c + t\mathbf{d}) \|\mathbf{d}\| dt,$$

where $t_0 = -\mathbf{f} \cdot c / \mathbf{f} \cdot \mathbf{d}$. Evaluating, we have

$$\tau(\mathbf{d}) = \frac{\alpha_0}{2} \|\mathbf{d}\| \frac{(\mathbf{f} \cdot c)^2}{\mathbf{f} \cdot \mathbf{d}}.$$

Chapter 9

1. If $\mathbf{p}_1 = \mathbf{p}_2$, then the enclosing sphere has the same center, and its radius is $\max(r_1, r_2)$. Otherwise, let $d = \|\mathbf{p}_2 - \mathbf{p}_1\|$ be the distance between the centers, and define $\mathbf{v} = (\mathbf{p}_2 - \mathbf{p}_1)/d$. Now let

$$a = \min(-r_1, d - r_2) \quad \text{and} \quad b = \max(r_1, d + r_2).$$

These represent the minimum and maximum extents of the spheres along a line through the two centers, where zero coincides with \mathbf{p}_1 . The center \mathcal{P} and radius R of the enclosing sphere are then given by

$$\mathcal{P} = \mathbf{p}_1 + \frac{a+b}{2} \mathbf{v} \quad \text{and} \quad R = \frac{b-a}{2}.$$

2. Suppose the equilateral triangle has sides of length one, and let its vertices be located at the coordinates $\mathbf{v}_1 = (-\frac{1}{2}, 0)$, $\mathbf{v}_2 = (\frac{1}{2}, 0)$, and $\mathbf{v}_3 = (0, \frac{\sqrt{3}}{2})$. The optimal bounding sphere has center \mathbf{p} and radius r given by

$$\mathbf{p} = \left(0, \frac{\sqrt{3}}{6} \right) \quad \text{and} \quad r = \frac{\sqrt{3}}{3}.$$

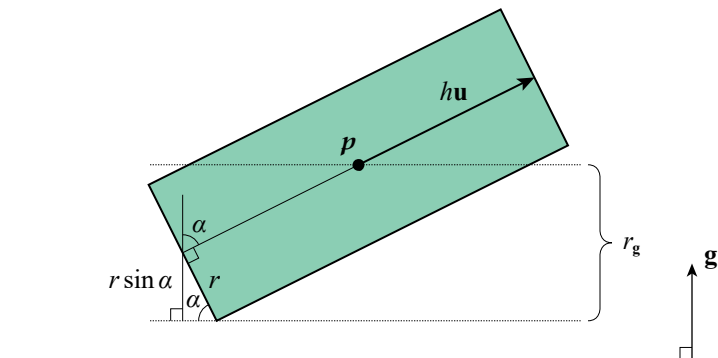
Without loss of generality, we assume that the vertices \mathbf{v}_1 and \mathbf{v}_2 are found to be farthest apart, and we establish an initial approximate bounding sphere having center at the origin and radius $\frac{1}{2}$. The vertex \mathbf{v}_3 is not inside this sphere. Expanding the initial sphere with Equation (9.7) produces the center \mathbf{p}' and radius r' given by

$$\mathbf{p}' = \left(0, \frac{\sqrt{3}-1}{4} \right) \quad \text{and} \quad r' = \frac{\sqrt{3}+1}{4}.$$

These both differ from the optimal center and radius by $(3 - \sqrt{3})/12 \approx 0.10566$.

3. $v^2 \geq (|h_x v_x| + |h_y v_y| + |h_z v_z| + r_{\max})^2$
 $\max(|v_x| - h_x, |v_y| - h_y, |v_z| - h_z) \geq r_{\max}$
4. The effective radius r_g is the sum of separate contributions from half the height of the cylinder and its actual radius. The contribution from the half height h is $|h\mathbf{g} \cdot \mathbf{u}|$. As shown in the figure, the contribution from the radius r is $r \sin \alpha$. Since $\cos \alpha = \mathbf{g} \cdot \mathbf{u}$, we can write this as $r\sqrt{1 - \cos^2 \alpha} = r\sqrt{1 - (\mathbf{g} \cdot \mathbf{u})^2}$. Then

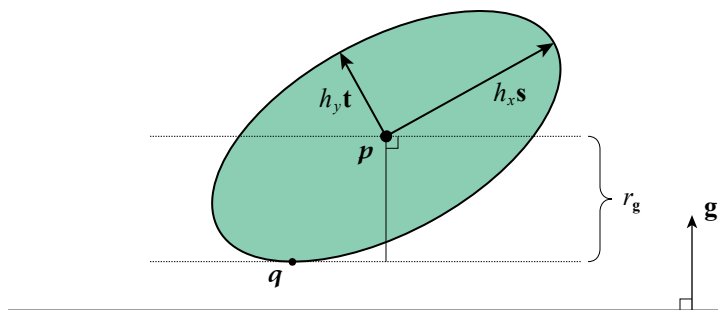
$$r_g = |h\mathbf{g} \cdot \mathbf{u}| + r\sqrt{1 - (\mathbf{g} \cdot \mathbf{u})^2}.$$



5. A point q on the surface of an ellipsoid with center p , shown in the figure, is given by

$$q = p + (h_x \cos \theta \sin \varphi) \mathbf{s} + (h_y \sin \theta \sin \varphi) \mathbf{t} + (h_z \cos \varphi) \mathbf{u},$$

where $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$.



We want to maximize $r_g = \mathbf{g} \cdot (\mathbf{q} - \mathbf{p})$, so we take derivatives with respect to θ and φ and set them to zero as follows:

$$\frac{\partial r_g}{\partial \theta} = (-h_x \sin \theta \sin \varphi)(\mathbf{g} \cdot \mathbf{s}) + (h_y \cos \theta \sin \varphi)(\mathbf{g} \cdot \mathbf{t}) = 0$$

$$\frac{\partial r_g}{\partial \varphi} = (h_x \cos \theta \cos \varphi)(\mathbf{g} \cdot \mathbf{s}) + (h_y \sin \theta \cos \varphi)(\mathbf{g} \cdot \mathbf{t}) - (h_z \sin \varphi)(\mathbf{g} \cdot \mathbf{u}) = 0.$$

Defining $x = h_x (\mathbf{g} \cdot \mathbf{s})$, $y = h_y (\mathbf{g} \cdot \mathbf{t})$, and $z = h_z (\mathbf{g} \cdot \mathbf{u})$, we can rewrite these as

$$\tan \theta = \frac{y}{x}$$

$$\tan \varphi = \frac{x}{z} \cos \theta + \frac{y}{z} \sin \theta.$$

The first of these is true for a triangle having legs of length x and y , so it must be the case that

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

Then $\tan \varphi$ can be written as

$$\tan \varphi = \frac{\sqrt{x^2 + y^2}}{z}.$$

Using the trigonometric identity $\cos \varphi = 1/\sqrt{1 + \tan^2 \varphi}$, we can now write

$$\cos \varphi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad \text{and} \quad \sin \varphi = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}.$$

Plugging all of these into the expression for r_g gives us

$$r_g = \sqrt{h_x^2 (\mathbf{g} \cdot \mathbf{s})^2 + h_y^2 (\mathbf{g} \cdot \mathbf{t})^2 + h_z^2 (\mathbf{g} \cdot \mathbf{u})^2}.$$

6. The minimum value of $\mathbf{g} \cdot \mathbf{z}$ for which the plane \mathbf{g} intersects the cone is $-\sin \alpha$. This means that points on the positive side of the plane are visible only when $\mathbf{g} \cdot \mathbf{z} > -\sin \alpha$.
7. This function generates a polyhedron corresponding to a spot light having projection distance g and aspect ratio s . The value of r_{\max} provides the distance to the far plane at the light's radius. The `Mlight` parameter specifies the object-space to world-space transformation for the light.

```
void BuildSpotPolyhedron(const Transform4D& Mlight, float g, float s, float rmax,
                        Polyhedron *polyhedron)
{
    polyhedron->vertexCount = 5; polyhedron->edgeCount = 8; polyhedron->faceCount = 5;

    // Generate vertex for the light position.
    polyhedron->vertex[0] = Mlight.GetTranslation();

    // Generate vertices for the far side.
    float y = rmax / g, x = y * s;
    polyhedron->vertex[1] = Mlight * Point3D(x, y, rmax);
    polyhedron->vertex[2] = Mlight * Point3D(x, -y, rmax);
    polyhedron->vertex[3] = Mlight * Point3D(-x, -y, rmax);
    polyhedron->vertex[4] = Mlight * Point3D(-x, y, rmax);

    // Generate lateral planes.
    Transform4D inverse = Inverse(Mlight);
    float mx = 1.0F / sqrt(g * g + s * s), my = 1.0F / sqrt(g * g + 1.0F);
    polyhedron->plane[0] = Plane(-g * mx, 0.0F, s * mx, 0.0F) * inverse;
    polyhedron->plane[1] = Plane(0.0F, g * my, my, 0.0F) * inverse;
    polyhedron->plane[2] = Plane(g * mx, 0.0F, s * mx, 0.0F) * inverse;
    polyhedron->plane[3] = Plane(0.0F, -g * my, my, 0.0F) * inverse;
}
```



```

// Generate far plane.
polyhedron->plane[4].Set(-Mlight[2], Dot(Mlight[2], Mlight[3]) + rmax);

// Generate all edges and lateral faces.
Edge *edge = polyhedron->edge; Face *face = polyhedron->face;
for (int32 i = 0; i < 4; i++, edge++, face++)
{
    edge[0].vertexIndex[0] = uint8(0);
    edge[0].vertexIndex[1] = uint8(i + 1);
    edge[0].faceIndex[0] = uint8(i);
    edge[0].faceIndex[1] = uint8((i - 1) & 3);

    edge[4].vertexIndex[0] = uint8(((i + 1) & 3) + 1);
    edge[4].vertexIndex[1] = uint8(i + 1);
    edge[4].faceIndex[0] = 4;
    edge[4].faceIndex[1] = uint8(i);

    face->edgeCount = 3;
    face->edgeIndex[0] = uint8(i);
    face->edgeIndex[1] = uint8((i + 1) & 3);
    face->edgeIndex[2] = uint8(i + 4);
}

// Generate far face.
face[0].edgeCount = 4;
face[0].edgeIndex[0] = 4; face[0].edgeIndex[1] = 5;
face[0].edgeIndex[2] = 6; face[0].edgeIndex[3] = 7;
}

```

8. Let $\mathbf{N} = \mathbf{M}_{\text{occluder}}^{-1}$. The six world-space bounding planes are given by $\mathbf{N}_0, \mathbf{N}_1, \mathbf{N}_2, s_x \mathbf{N}_3 - \mathbf{N}_0, s_y \mathbf{N}_3 - \mathbf{N}_1$, and $s_z \mathbf{N}_3 - \mathbf{N}_2$ (where the notation \mathbf{N}_i means row i of the matrix \mathbf{N} .)
9. The minimum value of φ is given by

$$\varphi = \sin^{-1} \left(-\frac{\alpha_0 z_0^2}{2\tau_{\max}} \right),$$

and this exists when $\alpha_0 z_0^2 / 2\tau_{\max} \leq 1$.

Chapter 10

1. Suppose that \mathbf{c} and \mathbf{p} lie at opposite ends of a diameter of the sphere. Then $t_1 = 0, t_2 = 1, \|\mathbf{v}\| = 2R, v^2 = 4R^2, p^2 = R^2$, and $\mathbf{p} \cdot \mathbf{v} = -2R^2$. Plugging these values into Equation (10.35) gives us

$$B = 2R \left[1 - \frac{R^2}{R^2} + \frac{2R^2}{R^2} - \frac{4R^2}{3R^2} \right] = \frac{4R}{3}.$$

$$\begin{aligned} 2. \quad \sum_{i=0}^n \left(\frac{2(1-m)}{n} i + m \right) &= m(n+1) + \frac{2(1-m)}{n} \sum_{i=1}^n i \\ &= m(n+1) + \left(\frac{2(1-m)}{n} \right) \left(\frac{n(n+1)}{2} \right) \\ &= m(n+1) + (1-m)(n+1) \\ &= n+1 \end{aligned}$$

3. First integrating over φ , we have

$$\begin{aligned} &\int_0^{2\pi} \int_0^\pi \frac{1-g^2}{4\pi(1+g^2-2g\cos\alpha)^{3/2}} \sin\alpha \, d\alpha \, d\varphi \\ &= \frac{1-g^2}{2} \int_0^\pi \frac{\sin\alpha}{(1+g^2-2g\cos\alpha)^{3/2}} \, d\alpha. \end{aligned}$$

Making the substitutions $u = 1 + g^2 - 2g \cos \alpha$ and $du = 2g \sin \alpha \, d\alpha$ then allows us to integrate over α to get

$$\begin{aligned} \frac{1-g^2}{4g} \int_{1+g^2-2g}^{1+g^2+2g} \frac{du}{u^{3/2}} &= \frac{g^2-1}{2g} \left[\frac{1}{\sqrt{u}} \right]_{(g-1)^2}^{(g+1)^2} \\ &= \frac{g^2-1}{2g} \left(\frac{1}{g+1} + \frac{1}{g-1} \right) \\ &= 1. \end{aligned}$$

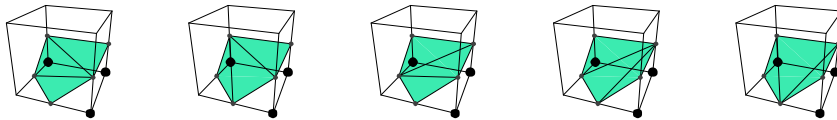
4. The following table lists the components of Equation (10.83) for the six possible orderings of a, b , and c .

Case	$d = \min(a, b)$	$e = \max(a, b)$	$f = \max(d, c)$	$\text{med}(a, b, c) = \min(f, e)$
$a < b < c$	a	b	c	b
$a < c < b$	a	b	c	c
$b < a < c$	b	a	c	a
$b < c < a$	b	a	c	c
$c < a < b$	a	b	a	a
$c < b < a$	b	a	b	b


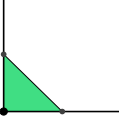
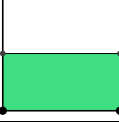
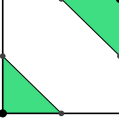
5. Most classes have multiple symmetry planes, and the table below lists corners for each class forming one of the possible planes when combined with the cube center using the numbering from Figure 10.32.

Class	Corners	Class	Corners
1	0, 2	9	0, 2
2	1, 4	10	1, 2
3	0, 2	11	0, 3
4	1, 2	14	0, 3
5	0, 3	15	0, 5
6	0, 2	16	0, 5
7	1, 2	17	1, 4
8	0, 3		

6. The five possible triangulations of class #5 are shown below.



7. Marching squares has 16 distinct cases that are partitioned into the four equivalence classes shown in the following table.

Class	Cases
	2
	8
	4
	2