Foundations
of
Game Engine Development

Volume 1
Mathematics

Solutions Guide

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Chapter 1

1. (a) $\mathbf{i}\cdot\mathbf{j}=0$, $\mathbf{j}\cdot\mathbf{k}=0$, and $\mathbf{k}\cdot\mathbf{i}=0$.
(b) $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$, $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$, $\mathbf{k}\times\mathbf{i}=\mathbf{j}$, and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$.
(c) $[\mathbf{i},\mathbf{j},\mathbf{k}]=1$.

2. $\|\mathbf{a}+\mathbf{b}\|^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=a^2+b^2+2\mathbf{a}\cdot\mathbf{b}$. For the dot product, we can say $\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta\le\|\mathbf{a}\|\|\mathbf{b}\|$, which means $\|\mathbf{a}+\mathbf{b}\|^2\le a^2+b^2+2\|\mathbf{a}\|\|\mathbf{b}\|$. The right side of this inequality is $(\|\mathbf{a}\|+\|\mathbf{b}\|)^2$, and taking square roots of both sides gives us the result.

3. Similar to previous exercise.

$$\|\mathbf{a}-\mathbf{b}\|^2=a^2+b^2-2\mathbf{a}\cdot\mathbf{b}\le a^2+b^2-2\|\mathbf{a}\|\|\mathbf{b}\|=(\|\mathbf{a}\|-\|\mathbf{b}\|)^2.$$

4. Expanding each term with the vector triple product identity, we have

$$\begin{aligned}&\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})\\&=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{c}(\mathbf{b}\cdot\mathbf{a})-\mathbf{a}(\mathbf{b}\cdot\mathbf{c})+\mathbf{a}(\mathbf{c}\cdot\mathbf{b})-\mathbf{b}(\mathbf{c}\cdot\mathbf{a}),\end{aligned}$$

and every term of the result cancels.

5. Note that $\mathbf{b}\times\mathbf{a}\times\mathbf{b}=\mathbf{b}\times(\mathbf{a}\times\mathbf{b})$. Applying the vector triple product identity, we have $\mathbf{b}\times\mathbf{a}\times\mathbf{b}=b^2\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\mathbf{b}$. Dividing by b^2 gives us

$$\frac{\mathbf{b}\times\mathbf{a}\times\mathbf{b}}{b^2}=\mathbf{a}-\frac{\mathbf{a}\cdot\mathbf{b}}{b^2}\mathbf{b}=\mathbf{a}-\mathbf{a}_{\parallel\mathbf{b}}=\mathbf{a}_{\perp\mathbf{b}}.$$

6. $$\mathbf{R}=\frac{1}{b^2}\begin{bmatrix}b_y^2+b_z^2 & -b_xb_y & -b_xb_z\\ -b_xb_y & b_x^2+b_z^2 & -b_yb_z\\ -b_xb_z & -b_yb_z & b_x^2+b_y^2\end{bmatrix}.$$

7. Expanding $\mathbf{v}_{\perp(\mathbf{a}\times\mathbf{b})}$, we have

$$\mathbf{v}_{\perp(\mathbf{a}\times\mathbf{b})}=\mathbf{v}-\frac{\mathbf{v}\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\times\mathbf{b})^2}(\mathbf{a}\times\mathbf{b}).$$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=0$, the entire last term disappears when we calculate $(\mathbf{v}_{\perp(\mathbf{a}\times\mathbf{b})})_{\parallel\mathbf{a}}$, and we are left with

$$\left(\mathbf{v}_{\perp(\mathbf{a}\times\mathbf{b})}\right)_{\|\mathbf{a}} = \frac{\mathbf{v}\cdot\mathbf{a}}{a^2}\mathbf{a} = \mathbf{v}_{\|\mathbf{a}}.$$

8. We want to show that $[(\mathbf{AB})\mathbf{C}]_{ij} = [\mathbf{A}(\mathbf{BC})]_{ij}$ for all $0 \le i < n$ and $0 \le j < q$. Starting with the left side, we have

$$\begin{aligned}
[(\mathbf{AB})\mathbf{C}]_{ij} &= \sum_{k=0}^{p-1} (\mathbf{AB})_{ik} C_{kj} \\
&= \sum_{k=0}^{p-1} \left[\left(\sum_{l=0}^{m-1} A_{il} B_{lk} \right) C_{kj} \right] \\
&= \sum_{l=0}^{m-1} \left[A_{il} \sum_{k=0}^{p-1} B_{lk} C_{kj} \right] \\
&= \sum_{l=0}^{m-1} A_{il} \, (\mathbf{BC})_{lj} \\
&= [\mathbf{A}(\mathbf{BC})]_{ij}.
\end{aligned}$$

9. For a 2×2 matrix, it's clear that the determinant is just the product of the two diagonal entries. To prove the general case by induction, assume that for any $k < n$, the determinant of a $k \times k$ matrix is equal to the product of its diagonal entries. If we calculate the determinant of an $n \times n$ matrix using expansion by minors along the last column, then the only nonzero term corresponds to the $(n-1, n-1)$ entry. This is multiplied by the determinant of the upper-left $(n-1)\times(n-1)$ portion of the matrix, which is known to be the product of its diagonal entries. Since we are now simply multiplying by one more diagonal entry, we have proven the general case.

10. In the product $t\mathbf{A}$, every entry of $\mathbf{A}$ is multiplied by t. This means that every factor in the Leibniz formula for the determinant is multiplied by t when $\mathbf{A}$ is replaced by $t\mathbf{A}$. Since every term has n factors, the full determinant is multiplied by a factor of t^n.

11. A matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular when the determinant $ad - bc$ is zero, so we need to find all cases for which $ad = bc$ with a, b, c, and d being either 0 or 1. Of the 16 possible 2×2 matrices, the following 10 are singular.

$$\begin{bmatrix}0&0\\0&0\end{bmatrix}\quad\begin{bmatrix}1&0\\0&0\end{bmatrix}\quad\begin{bmatrix}0&1\\0&0\end{bmatrix}\quad\begin{bmatrix}0&0\\1&0\end{bmatrix}\quad\begin{bmatrix}0&0\\0&1\end{bmatrix}$$
$$\begin{bmatrix}1&1\\0&0\end{bmatrix}\quad\begin{bmatrix}0&1\\0&1\end{bmatrix}\quad\begin{bmatrix}0&0\\1&1\end{bmatrix}\quad\begin{bmatrix}1&0\\1&0\end{bmatrix}\quad\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

12. Multiply both sides of the equation $\mathbf{LM} = \mathbf{I}$ on the right by $\mathbf{R}$. Then $\mathbf{LMR} = \mathbf{R}$, but $\mathbf{MR} = \mathbf{I}$, so $\mathbf{L} = \mathbf{R}$.

13. The following matrix is the inverse of the elementary matrix **E** that multiplies row r by t.

$$\mathbf{E}^{-1} = \begin{bmatrix}1&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&&\vdots\\ 0&\cdots&1/t&\cdots&0\\ \vdots&&\vdots&\ddots&\vdots\\ 0&\cdots&0&\cdots&1\end{bmatrix}\begin{matrix}\\ \\ \leftarrow \text{row } r\\ \\ \\ \end{matrix}$$

The following matrix is the inverse of the elementary matrix **E** that exchanges rows r and s. (In this case, **E** is its own inverse.)

$$\mathbf{E}^{-1} = \begin{bmatrix}1&\cdots&0&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&&\vdots&&\vdots\\ 0&\cdots&0&\cdots&1&\cdots&0\\ \vdots&&\vdots&\ddots&\vdots&&\vdots\\ 0&\cdots&1&\cdots&0&\cdots&0\\ \vdots&&\vdots&&\vdots&\ddots&\vdots\\ 0&\cdots&0&\cdots&0&\cdots&1\end{bmatrix}\begin{matrix}\\ \\ \leftarrow \text{row } r\\ \\ \leftarrow \text{row } s\\ \\ \\ \end{matrix}$$

The following matrix is the inverse of the elementary matrix **E** that adds row s multiplied by the scalar value t to row r.

$$\mathbf{E}^{-1} = \begin{array}{c} \text{column } s \\ \downarrow \\ \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & -t & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \leftarrow \text{row } r \\ \\ \leftarrow \text{row } s \\ \\ \\ \end{array}$$

14. When the permutation τ is applied, the Leibniz formula becomes

$$\det(\mathbf{M}) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\tau\sigma) \prod_{k=0}^{n-1} M_{k,\tau(\sigma(k))} \right).$$

The sign of the permutation τ is -1 because it is a single transposition. Since $\operatorname{sgn}(\tau\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)$, every term in the summation is negated.

15. Suppose that $\mathbf{M}$ is an $n \times n$ matrix and that row r of $\mathbf{M}$ is all zeros. Let $\mathbf{N}$ be any $n \times n$ matrix. The (r, r) entry of the product $\mathbf{MN}$ is given by

$$(\mathbf{MN})_{rr} = \sum_{k=0}^{n-1} M_{rk} N_{kr},$$

but each $M_{rk} = 0$, so it's impossible to produce anything other than zero. Since a one is needed in the (r, r) entry of the identity matrix, $\mathbf{M}$ cannot be invertible.

16. First, assume that $\mathbf{M}$ is invertible. Then $\mathbf{I} = \mathbf{I}^{\mathrm{T}} = (\mathbf{M}\mathbf{M}^{-1})^{\mathrm{T}} = (\mathbf{M}^{-1})^{\mathrm{T}} \mathbf{M}^{\mathrm{T}}$, and this shows that $(\mathbf{M}^{-1})^{\mathrm{T}}$ is the inverse of $\mathbf{M}^{\mathrm{T}}$. Second, assume that $\mathbf{M}^{\mathrm{T}}$ is invertible. Then $\mathbf{I}^{\mathrm{T}} = \mathbf{I} = (\mathbf{M}^{\mathrm{T}})^{-1} \mathbf{M}^{\mathrm{T}} = \left[\mathbf{M}\left((\mathbf{M}^{\mathrm{T}})^{-1}\right)^{\mathrm{T}}\right]^{\mathrm{T}}$, and this shows that $\left((\mathbf{M}^{\mathrm{T}})^{-1}\right)^{\mathrm{T}}$ is the inverse of $\mathbf{M}$. Since we know that $(\mathbf{M}^{-1})^{\mathrm{T}}$ is the inverse of $\mathbf{M}^{\mathrm{T}}$ from the first part, and we know that $(\mathbf{M}^{\mathrm{T}})^{-1}$ is the inverse of $\mathbf{M}^{\mathrm{T}}$ from the second part, it must be the case that $(\mathbf{M}^{\mathrm{T}})^{-1} = (\mathbf{M}^{-1})^{\mathrm{T}}$.

17. The (i, j) entry of the product $\operatorname{adj}(\mathbf{M})\mathbf{M}$ is given by

$$(\operatorname{adj}(\mathbf{M})\mathbf{M})_{ij} = \sum_{k=0}^{n-1} C_{ki}(\mathbf{M}) M_{kj}.$$

When $i = j$, the summation is equal to the determinant of $\mathbf{M}$. When $i \neq j$, the summation is equal to the determinant of $\mathbf{M}$ if it were modified so that the entries in column j were replaced by the entries in column i. This matrix has two identical columns and must therefore have a determinant of zero. Thus, the entries on the main diagonal of $\operatorname{adj}(\mathbf{M})\mathbf{M}$ are all $\det(\mathbf{M})$, and all other entries are zero. A similar argument can be applied to $\mathbf{M}\operatorname{adj}(\mathbf{M})$.

18. Multiplying the first two matrices gives us

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{b}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_n + \mathbf{a} \otimes \mathbf{b} & \mathbf{a} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n + \mathbf{a} \otimes \mathbf{b} & \mathbf{a} \\ \mathbf{b}^{\mathrm{T}} + \mathbf{b}^{\mathrm{T}}(\mathbf{a} \otimes \mathbf{b}) & \mathbf{b}^{\mathrm{T}}\mathbf{a} + 1 \end{bmatrix}.$$

Now multiplying by the third matrix produces

$$\begin{bmatrix} \mathbf{I}_n + \mathbf{a} \otimes \mathbf{b} & \mathbf{a} \\ \mathbf{b}^{\mathrm{T}} + \mathbf{b}^{\mathrm{T}}(\mathbf{a} \otimes \mathbf{b}) & \mathbf{b}^{\mathrm{T}}\mathbf{a} + 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{b}^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{a} \\ \mathbf{0} & 1 + \mathbf{a} \cdot \mathbf{b} \end{bmatrix},$$

where we have used the facts that $\mathbf{ab}^{\mathrm{T}} = \mathbf{a} \otimes \mathbf{b}$ and $\mathbf{b}^{\mathrm{T}}\mathbf{a} = \mathbf{a} \cdot \mathbf{b}$. The determinants of both sides of the original matrix identity are easily calculated because each matrix contains a row or column that is all zeros except for the entry in the lower-right corner. The determinant of the left side is the product of the determinants of the three matrices, and the determinants of the first and third matrices are just one, so the determinant of the whole left side is simply the determinant of the $n \times n$ matrix $\mathbf{I}_n + \mathbf{a} \otimes \mathbf{b}$. The determinant of the right side is equal to $(1 + \mathbf{a} \cdot \mathbf{b})\det(\mathbf{I}_n)$. Therefore, $\det(\mathbf{I}_n + \mathbf{a} \otimes \mathbf{b}) = 1 + \mathbf{a} \cdot \mathbf{b}$.

19. The diagonal entries of $\mathbf{M}^{-1}\mathbf{M}$ are given by

$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{00} = \frac{(\mathbf{b} \times \mathbf{v} + y\mathbf{t}) \cdot \mathbf{a} - (\mathbf{b} \cdot \mathbf{t})x}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}$$
$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{11} = \frac{(\mathbf{v} \times \mathbf{a} - x\mathbf{t}) \cdot \mathbf{b} + (\mathbf{a} \cdot \mathbf{t})y}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}$$
$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{22} = \frac{(\mathbf{d} \times \mathbf{u} + w\mathbf{s}) \cdot \mathbf{c} - (\mathbf{d} \cdot \mathbf{s})z}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}$$
$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{33} = \frac{(\mathbf{u} \times \mathbf{c} - z\mathbf{s}) \cdot \mathbf{d} + (\mathbf{c} \cdot \mathbf{s})w}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}$$

A little regrouping in the numerators gives us

$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{00} = \frac{[\mathbf{b}, \mathbf{v}, \mathbf{a}] + \mathbf{t} \cdot (y\mathbf{a} - x\mathbf{b})}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}$$
$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{11} = \frac{[\mathbf{v}, \mathbf{a}, \mathbf{b}] + \mathbf{t} \cdot (y\mathbf{a} - x\mathbf{b})}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}$$
$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{22} = \frac{[\mathbf{d}, \mathbf{u}, \mathbf{c}] + \mathbf{s} \cdot (w\mathbf{c} - z\mathbf{d})}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}$$
$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{33} = \frac{[\mathbf{u}, \mathbf{c}, \mathbf{d}] + \mathbf{s} \cdot (w\mathbf{c} - z\mathbf{d})}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}.$$

The scalar triple products can each be permuted to give either $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ or $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{u}$, and then it is clear that the numerators and denominators are equal, so the diagonal entries are all ones.

Off the diagonal, we choose the $(1, 0)$ entry, which is calculated as

$$\left(\mathbf{M}^{-1}\mathbf{M}\right)_{10} = \frac{(\mathbf{v} \times \mathbf{a} - x\mathbf{t}) \cdot \mathbf{a} + (\mathbf{a} \cdot \mathbf{t})\, x}{\mathbf{s} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u}}.$$

The term $(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}$ is zero, and the remaining two terms in the numerator cancel, producing a zero for the $(1, 0)$ entry.

20. The x component of the left side is

$$[(a_y b_z - a_z b_y)\, c_x + (a_z b_x - a_x b_z)\, c_y + (a_x b_y - a_y b_x)\, c_z]\, d_x,$$

and the x component of the right side is

$$(a_x d_x + a_y d_y + a_z d_z)(b_y c_z - b_z c_y) + (b_x d_x + b_y d_y + b_z d_z)(c_y a_z - c_z a_y)$$
$$+ (c_x d_x + c_y d_y + c_z d_z)(a_y b_z - a_z b_y).$$

Collecting the terms on the right side containing the factor d_x gives us

$$[a_x (b_y c_z - b_z c_y) + b_x (c_y a_z - c_z a_y) + c_x (a_y b_z - a_z b_y)]\, d_x,$$

and each of these terms matches a term from the left side. We just need to show that the remaining terms on the right side sum to zero. Those terms are

$$(a_y d_y + a_z d_z)(b_y c_z - b_z c_y) + (b_y d_y + b_z d_z)(c_y a_z - c_z a_y)$$
$$+ (c_y d_y + c_z d_z)(a_y b_z - a_z b_y),$$

and multiplying everything out produces

$$a_y b_y c_z d_y - a_y b_z c_y d_y + a_z b_y c_z d_z - a_z b_z c_y d_z + a_z b_y c_y d_y - a_y b_y c_z d_y$$
$$+ a_z b_z c_y d_z - a_y b_z c_z d_z + a_y b_z c_y d_y - a_z b_y c_y d_y + a_y b_z c_z d_z - a_z b_y c_z d_z.$$

Every term cancels, so the x components of both sides are equal.

Chapter 2

1. A matrix $\mathbf{M}$ is an involution if $\mathbf{M}^2 = \mathbf{I}$. The determinant of $\mathbf{M}$ must therefore square to one, which leaves only two possibilities, $+1$ or -1.

2. Since $\mathbf{A}$ and $\mathbf{B}$ for orthogonal, we know $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$ and $\mathbf{B}^{-1} = \mathbf{B}^{\mathrm{T}}$. We then have

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = (\mathbf{AB})^{\mathrm{T}}.$$

Therefore, $\mathbf{AB}$ is orthogonal.

3. Let $\mathbf{M}$ be a symmetric matrix that is also an involution. Then $\mathbf{M}^2 = \mathbf{I}$. Since $\mathbf{M}$ is symmetric, $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$, so we can write $\mathbf{MM}^{\mathrm{T}} = \mathbf{I}$, and this shows that $\mathbf{M}$ is orthogonal. Now suppose that $\mathbf{M}$ is an orthogonal matrix that is also an involution. Since $\mathbf{M}$ is orthogonal, $\mathbf{M}^{-1} = \mathbf{M}^{\mathrm{T}}$, and since $\mathbf{M}$ is an involution, $\mathbf{M}^{-1} = \mathbf{M}$. Thus, $\mathbf{M}^{\mathrm{T}} = \mathbf{M}$, and this shows that $\mathbf{M}$ is symmetric.

4. $\mathbf{N} = \begin{bmatrix} 1/a^2 & 0 & 0 \\ 0 & 1/b^2 & 0 \\ 0 & 0 & 1/c^2 \end{bmatrix}.$

5. $\begin{bmatrix} (1-s)a_x^2 + s & (1-s)a_x a_y & (1-s)a_x a_z \\ (1-s)a_x a_y & (1-s)a_y^2 + s & (1-s)a_y a_z \\ (1-s)a_x a_z & (1-s)a_y a_z & (1-s)a_z^2 + s \end{bmatrix}.$

6. (a) $\mathbf{M}_{\text{skew}}(\theta, \mathbf{i}, \mathbf{k}) = \begin{bmatrix} 1 & 0 & \tan\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(b) $\mathbf{M}_{\text{skew}}(\theta, \mathbf{j}, \mathbf{k}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \tan\theta \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $\mathbf{M}_{\text{skew}}(\theta, \mathbf{j}, \mathbf{i}) = \begin{bmatrix} 1 & 0 & 0 \\ \tan\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7. The fourth row of $\mathbf{HG}$ is given by $H_{30}\mathbf{G}_0 + H_{31}\mathbf{G}_1 + H_{32}\mathbf{G}_2 + H_{33}\mathbf{G}_3$, where $\mathbf{G}_i$ represents row i of $\mathbf{G}$. Since $H_{30} = H_{31} = H_{32} = 0$ and $H_{33} = 1$, the fourth row of the product $\mathbf{HG}$ is equal to the fourth row of $\mathbf{G}$, which is $[0 \quad 0 \quad 0 \quad 1]$.

8. $\begin{bmatrix} \mathbf{M} & c - \mathbf{M}c \\ \mathbf{0} & 1 \end{bmatrix}$.

9. Let $\mathbf{q}_1 = \mathbf{v}_1 + s_1$ and $\mathbf{q}_2 = \mathbf{v}_2 + s_2$. Then

$$\|\mathbf{q}_1\mathbf{q}_2\|^2 = (\mathbf{v}_1 \times \mathbf{v}_2 + s_1\mathbf{v}_2 + s_2\mathbf{v}_1)^2 + (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2,$$

and this expands to

$$\|\mathbf{q}_1\mathbf{q}_2\|^2 = (\mathbf{v}_1 \times \mathbf{v}_2)^2 + s_1^2 v_2^2 + s_2^2 v_1^2 + s_1^2 s_2^2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2.$$

After applying Lagrange's identity to the cross product and dot product, this becomes

$$\|\mathbf{q}_1\mathbf{q}_2\|^2 = v_1^2 v_2^2 + s_1^2 v_2^2 + s_2^2 v_1^2 + s_1^2 s_2^2 = \left(v_1^2 + s_1^2\right)\left(v_2^2 + s_2^2\right) = \|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2.$$

Taking square roots of both sides shows that the magnitude of the product is equal to the product of the magnitudes.

10. $f(\mathbf{q}) = \dfrac{\mathbf{q} + \mathbf{q}^*}{2}$ and $g(\mathbf{q}) = \dfrac{\mathbf{q} - \mathbf{q}^*}{2}$.

11. The magnitude of $\mathbf{q}$ is given by

$$\|\mathbf{q}\|^2 = \left(\sin\frac{\theta}{2}\right)^2 a^2 + \left(\cos\frac{\theta}{2}\right)^2.$$

We know that $a^2 = 1$, so we are left with the sum of a squared sine and squared cosine of the same angle, which is always one.

12. The quaternions $\mathbf{q} = i$, $\mathbf{q} = j$, and $\mathbf{q} = k$ perform 180-degree rotations about the x, y, and z axes, respectively.

13. For the x axis, $\mathbf{q} = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2}$. For the y axis, $\mathbf{q} = \frac{\sqrt{2}}{2} j + \frac{\sqrt{2}}{2}$. For the z axis, $\mathbf{q} = \frac{\sqrt{2}}{2} k + \frac{\sqrt{2}}{2}$.

14. It's required that $c = \cos(\theta/2)$. Applying the relationship $\cos\theta = \mathbf{v}_1 \cdot \mathbf{v}_2$ and the trigonometric identity $\cos^2(\theta/2) = (1 + \cos\theta)/2$, we find that

$$c = \sqrt{\frac{1 + \mathbf{v}_1 \cdot \mathbf{v}_2}{2}}.$$

We must also have $s\mathbf{a} = \sin(\theta/2)(\mathbf{v}_1 \times \mathbf{v}_2)/\sin\theta$. This can be simplified using the identity $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, and the quaternion can be written as

$$\mathbf{q} = (1/2c)(\mathbf{v}_1 \times \mathbf{v}_2) + c.$$

Chapter 3

1. 3

2. The closest point is given by $\boldsymbol{p} + (\boldsymbol{q} - \boldsymbol{p})_{\parallel \mathbf{v}}$, which corresponds to the parameter $t = \dfrac{(\boldsymbol{q} - \boldsymbol{p}) \cdot \mathbf{v}}{v^2}$.

3. The function f expands to $f(t_1, t_2) = (\boldsymbol{p}_2 + t_2\mathbf{v}_2 - \boldsymbol{p}_1 - t_1\mathbf{v}_1)^2$, and its partial derivatives are

$$\frac{\partial f}{\partial t_1} = 2\left(\boldsymbol{p}_1 \cdot \mathbf{v}_1 - \boldsymbol{p}_2 \cdot \mathbf{v}_1 + t_1 v_1^2 - t_2 \mathbf{v}_1 \cdot \mathbf{v}_2\right)$$
$$\frac{\partial f}{\partial t_2} = 2\left(\boldsymbol{p}_2 \cdot \mathbf{v}_2 - \boldsymbol{p}_1 \cdot \mathbf{v}_2 + t_2 v_2^2 - t_1 \mathbf{v}_1 \cdot \mathbf{v}_2\right).$$

Simultaneously setting these to zero (and dropping the factors of two) gives us the system

$$\begin{bmatrix} v_1^2 & -\mathbf{v}_1 \cdot \mathbf{v}_2 \\ -\mathbf{v}_1 \cdot \mathbf{v}_2 & v_2^2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} (\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \mathbf{v}_1 \\ (\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot \mathbf{v}_2 \end{bmatrix}.$$

When we negate the bottom row and solve for t_1 and t_2, the result is equivalent to Equation (3.24).

4. The line where the two planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$ intersect is given by $\{\mathbf{n}_1 \times \mathbf{n}_2 \mid d_1\mathbf{n}_2 - d_2\mathbf{n}_1\}$, and the homogeneous point closest to the origin on this line is given by

$$\left(\mathbf{v} \times (d_1\mathbf{n}_2 - d_2\mathbf{n}_1) \;\middle|\; v^2\right),$$

where $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. Distributing the cross product and dividing by the w coordinate yields Equation (3.42).

5. $\mathbf{n}' = \mathbf{n}$ and $d' = d - \mathbf{n} \cdot \mathbf{t}$.

6. $\mathbf{v}' = \mathbf{v}$ and $\mathbf{m}' = \mathbf{m} + \mathbf{t} \times \mathbf{v}$.

7. By formula F in Table 3.1, the line $\{\mathbf{v} \mid \mathbf{m}\}$ intersects the plane $[\mathbf{v} \mid 0]$ at the homogeneous point $(\mathbf{m} \times \mathbf{v} \mid -v^2)$, which is equivalent to $(\mathbf{v} \times \mathbf{m} \mid v^2)$.

8. $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$, and either $d = \mathbf{v}_1 \cdot \mathbf{m}_2$ or $d = -\mathbf{v}_2 \cdot \mathbf{m}_1$.

9. $\mathbf{n} = v^2(\mathbf{m}_2 - \mathbf{m}_1)$ and $d = [\mathbf{v}, \mathbf{m}_1, \mathbf{m}_2]$.

10. $d = \dfrac{\|\mathbf{v} \times (\mathbf{m}_2 - \mathbf{m}_1)\|}{v^2}$.

11. $[(\mathbf{v}_p \times \mathbf{v}_q) \cdot (\boldsymbol{b} - \boldsymbol{a})]t^2 + [(\boldsymbol{p}_0 \times \mathbf{v}_q - \boldsymbol{q}_0 \times \mathbf{v}_p) \cdot (\boldsymbol{b} - \boldsymbol{a}) + (\mathbf{v}_p - \mathbf{v}_q) \cdot (\boldsymbol{a} \times \boldsymbol{b})]t$
$+ (\boldsymbol{p}_0 \times \boldsymbol{q}_0) \cdot (\boldsymbol{b} - \boldsymbol{a}) + (\boldsymbol{p}_0 - \boldsymbol{q}_0) \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0.$

12. Set $\mathbf{f} = [\mathbf{v}_2 \times \mathbf{u} \mid -\mathbf{u} \cdot \mathbf{m}_2]$. This intersects the line $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ at the homogeneous point $(\mathbf{p} \mid w) = (\mathbf{m}_1 \times (\mathbf{v}_2 \times \mathbf{u}) - (\mathbf{u} \cdot \mathbf{m}_2)\mathbf{v}_1 \mid -(\mathbf{v}_2 \times \mathbf{u}) \cdot \mathbf{v}_1)$. The w coordinate is equivalent to $(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - v_1^2 v_2^2$. The vector triple product and the identity given by Exercise 20 in Chapter 1 can be applied to produce the following more symmetric formula for $\mathbf{p}$:

$$\begin{aligned}\mathbf{p} = {} & (\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_1 \times \mathbf{m}_2 + \mathbf{v}_2 \times \mathbf{m}_1) - v_1^2(\mathbf{v}_2 \times \mathbf{m}_2) - v_2^2(\mathbf{v}_1 \times \mathbf{m}_1) \\ & - (\mathbf{m}_2 \cdot \mathbf{v}_1)(\mathbf{v}_1 \times \mathbf{v}_2)\end{aligned}$$

Chapter 4

1. The area of region A is $(a_x - b_x)(b_y - a_y)$, the area of region B is $\frac{1}{2} b_x (b_y - a_y)$, and the area of region C is $\frac{1}{2} a_y (a_x - b_x)$. The total area of the parallelogram spanned by **a** and **b** is $A + 2B + 2C$, which simplifies to the quantity $a_x b_y - a_y b_x$.

2. Assuming the products are not zero, $\overline{\mathbf{A}} \wedge \overline{\mathbf{B}} \neq \underline{\mathbf{A}} \wedge \underline{\mathbf{B}}$ when n is even and $\operatorname{gr}(\mathbf{A}) + \operatorname{gr}(\mathbf{B})$ is odd.

3. Suppose that **A** and **B** are both $(n-1)$-blades. The set of basis vectors composing **A** and **B** can differ by at most one element because it would otherwise require that we have more than n basis vectors. Therefore, we can write $\mathbf{A} = \mathbf{C} \wedge \mathbf{v}_1$ and $\mathbf{B} = \mathbf{C} \wedge \mathbf{v}_2$, where **C** is an $(n-2)$-blade representing the common factor between **A** and **B**, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors. The sum $\mathbf{A} + \mathbf{B}$ is then $\mathbf{C} \wedge (\mathbf{v}_1 + \mathbf{v}_2)$, and this is an $(n-1)$-blade because $\mathbf{v}_1 + \mathbf{v}_2$ is just a vector. Since any $(n-1)$-vector can be decomposed into a sum of $(n-1)$-blades, this proves that all $(n-1)$-vectors must be $(n-1)$-blades.

4. First assume that **A** is a 2-blade. Then $\mathbf{A} = \mathbf{a} \wedge \mathbf{b}$ for some vectors **a** and **b**, and clearly, $\mathbf{A} \wedge \mathbf{A} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{a} \wedge \mathbf{b} = 0$.

Now assume that $\mathbf{A} \wedge \mathbf{A} = 0$, and suppose that **A** is not a 2-blade. Then **A** can be written as $\mathbf{A} = \mathbf{a} \wedge \mathbf{b} + \mathbf{B}$, where **a** and **b** are vectors, and **B** is a 2-vector such that $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{B} \neq 0$. That is, the basis elements of $\mathbf{a} \wedge \mathbf{b}$ are independent of the basis elements of **B**, and the fact that the dimensionality is at least four guarantees that enough independent basis vectors are available. This means that $\mathbf{A} \wedge \mathbf{A}$ contains a nonzero term, which is a contradiction, so it must be the case that **A** is a 2-blade.

5. Assume $\overline{\mathbf{A} \wedge \mathbf{B}} = \overline{\mathbf{A}} \vee \overline{\mathbf{B}}$. Changing all right complements to left complements, we have

$$\underline{\mathbf{A} \wedge \mathbf{B}} = (-1)^{(\operatorname{gr}(\mathbf{A}) + \operatorname{gr}(\mathbf{B}))(n - \operatorname{gr}(\mathbf{A}) - \operatorname{gr}(\mathbf{B}))}\, \overline{\mathbf{A} \wedge \mathbf{B}}$$

and

$$\underline{\mathbf{A}} \vee \underline{\mathbf{B}} = (-1)^{\operatorname{gr}(\mathbf{A})(n - \operatorname{gr}(\mathbf{A}))}\, \overline{\mathbf{A}} \vee (-1)^{\operatorname{gr}(\mathbf{B})(n - \operatorname{gr}(\mathbf{B}))}\, \overline{\mathbf{B}}.$$

The powers of -1 in the first and second lines differ by $2\,\mathrm{gr}(\mathbf{A})\,\mathrm{gr}(\mathbf{B})$, which has no effect because it is always even, so $\underline{\mathbf{A} \wedge \mathbf{B}} = \underline{\mathbf{A}} \vee \underline{\mathbf{B}}$. The same proof applies to $\overline{\mathbf{A} \vee \mathbf{B}} = \overline{\mathbf{A}} \wedge \overline{\mathbf{B}}$ as well as the reverse implications.

6. We first show that $\mathbf{A} \vee \overline{\mathbf{B}} = \underline{\mathbf{A}} \vee \mathbf{B}$. Each component of $\mathbf{A}$ and $\mathbf{B}$ has the form $\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k}$. In each of the antiwedge products $\mathbf{A} \vee \overline{\mathbf{B}}$ and $\underline{\mathbf{A}} \vee \mathbf{B}$, the only nonzero products between components are those that have identical basis elements in $\mathbf{A}$ and $\mathbf{B}$ because for all other pairings, the complement of one would exclude a basis vector that is also excluded by the other. For any basis element $\mathbf{C}$ shared by $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{C} \vee \overline{\mathbf{C}} = 1$ and $\underline{\mathbf{C}} \vee \mathbf{C} = 1$, so $\mathbf{A} \vee \overline{\mathbf{B}}$ and $\underline{\mathbf{A}} \vee \mathbf{B}$ are both equal to the same sum of the products of coefficients of like components.

Now we only need to show that $\mathbf{B} \vee \overline{\mathbf{A}} = \underline{\mathbf{A}} \vee \mathbf{B}$. By reversing the order of the factors, we have

$$\overline{\mathbf{A}} \vee \mathbf{B} = (-1)^{k(n-k)}\, \mathbf{B} \vee \overline{\mathbf{A}}.$$

Changing the right complement to left complement gives us

$$\underline{\mathbf{A}} \vee \mathbf{B} = (-1)^{k(n-k)} (-1)^{k(n-k)}\, \mathbf{B} \vee \overline{\mathbf{A}}.$$

The total power of -1 is always even, so the equality holds.

7. By definition, $\mathbf{A} \dashv (\mathbf{B} \dashv \mathbf{C}) = \underline{\mathbf{A}} \vee (\underline{\mathbf{B}} \vee \mathbf{C})$. By the associativity of the antiwedge product, $\underline{\mathbf{A}} \vee (\underline{\mathbf{B}} \vee \mathbf{C}) = (\underline{\mathbf{A}} \vee \underline{\mathbf{B}}) \vee \mathbf{C}$. Then

$$(\underline{\mathbf{A}} \vee \underline{\mathbf{B}}) \vee \mathbf{C} = \underline{\mathbf{A} \wedge \mathbf{B}} \vee \mathbf{C} = (\mathbf{A} \wedge \mathbf{B}) \dashv \mathbf{C}.$$

8. The dot product between (L_{vx}, L_{vy}, L_{vz}) and (L_{mx}, L_{my}, L_{mz}) is

$$\begin{aligned}&(a_w b_x - a_x b_w)(a_y b_z - a_z b_y) + (a_w b_y - a_y b_w)(a_z b_x - a_x b_z)\\&\quad + (a_w b_z - a_z b_w)(a_x b_y - a_y b_x).\end{aligned}$$

When multiplied out, every term cancels, so the dot product is zero, and the two vectors are therefore orthogonal.

9. Direct multiplication produces the following.

$$\begin{aligned}
\boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} = [& (q_x - p_x)\mathbf{e}_{41} + (q_y - p_y)\mathbf{e}_{42} + (q_z - p_z)\mathbf{e}_{43} \\
& + (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}] \wedge \boldsymbol{r} \\
= & (q_y r_z - p_y r_z + p_z r_y - q_z r_y + p_y q_z - p_z q_y)\mathbf{e}_{234} \\
& + (p_x r_z - q_x r_z + q_z r_x - p_z r_x + p_z q_x - p_x q_z)\mathbf{e}_{314} \\
& + (q_x r_y - p_x r_y + p_y r_x - q_y r_x + p_x q_y - p_y q_x)\mathbf{e}_{124} \\
& + (p_y q_z r_x - p_z q_y r_x + p_z q_x r_y - p_x q_z r_y + p_x q_y r_z - p_y q_x r_z)\mathbf{e}_{123} \\
= & (\mathbf{p} \wedge \mathbf{q} + \mathbf{q} \wedge \mathbf{r} + \mathbf{r} \wedge \mathbf{p}) \wedge \mathbf{e}_4 - (\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})\overline{\mathbf{e}}_4
\end{aligned}$$

10. $\boldsymbol{p} \vee \overline{\mathbf{e}}_4 = p_w$, $\mathbf{L} \vee \overline{\mathbf{e}}_4 = L_{vx}\mathbf{e}_1 + L_{vy}\mathbf{e}_2 + L_{vz}\mathbf{e}_3$, and $\mathbf{f} \vee \overline{\mathbf{e}}_4 = f_x\mathbf{e}_{23} + f_y\mathbf{e}_{31} + f_z\mathbf{e}_{12}$.

11. $\dfrac{-1}{\underline{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} \wedge \mathbf{f} \wedge \mathbf{g} \wedge \mathbf{h}}}\left(\underline{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{f} \wedge \mathbf{g} \wedge \mathbf{h}}\right)$.

12. $[(1+\mathbf{v})/2]^2 = \frac{1}{4}(1 + 2\mathbf{v} + \mathbf{v} \wedge \mathbf{v} + \mathbf{v} \cdot \mathbf{v})$. Since $\mathbf{v} \wedge \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{v} = 1$ (because $\mathbf{v}$ has unit length), the right side simplifies to $(1+\mathbf{v})/2$.

13. $f(k) = \dfrac{k(k-1)}{2}$.

14. Let $\mathbf{p}$ be the point closest to the origin on the line $\mathbf{L}$. Then we can write $\mathbf{m}$ as $\mathbf{m} = \mathbf{p} \wedge (\mathbf{p} + \mathbf{v}) = \mathbf{p} \wedge \mathbf{v}$ because $\mathbf{p}$ and $\mathbf{p} + \mathbf{v}$ are two points on the line. We know that $\mathbf{p}$ and $\mathbf{v}$ are perpendicular, so $\mathbf{p} \cdot \mathbf{v} = 0$, and we can write $\mathbf{m} = \mathbf{pv}$ using the geometric product. Division by $\mathbf{v}$ gives us $\mathbf{p} = \mathbf{m}/\mathbf{v}$.

15. Let $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \cdots$. The product $\mathbf{AB}$ consists of terms $\mathbf{A}_i\mathbf{B}_j$, where $\mathbf{A}_i$ and $\mathbf{B}_j$ have even grade and are thus composed of terms having the form $\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k}$, where k is even. When $\mathbf{A}_i$ and $\mathbf{B}_j$ are multiplied together, the grade of the result is the sum of the grades of $\mathbf{A}_i$ and $\mathbf{B}_j$ less twice the number of basis vectors that they have in common because one vector is eliminated from $\mathbf{A}_i$, and one vector is eliminated from $\mathbf{B}_j$. The resulting grade thus always remains even, and the subalgebra is closed.

16. The even subalgebra of the three-dimensional geometric algebra has the basis elements $\{\pm 1, \pm\mathbf{e}_{23}, \pm\mathbf{e}_{31}, \pm\mathbf{e}_{12}\}$. The two-dimensional Clifford algebra has the basis elements $\{\pm 1, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_{12}\}$. In the 3D case, let $\mathbf{x} = -\mathbf{e}_{23}$, $\mathbf{y} = -\mathbf{e}_{31}$, and $\mathbf{z} = -\mathbf{e}_{12}$. In the 2D case, let $\mathbf{x} = \mathbf{e}_1$, $\mathbf{y} = \mathbf{e}_2$, and $\mathbf{z} = \mathbf{e}_{12}$. Then in both cases, we have $x^2 = y^2 = z^2 = -1$, $\mathbf{xy} = \mathbf{z}$, $\mathbf{yz} = \mathbf{x}$, and $\mathbf{zx} = \mathbf{y}$. This is sufficient to establish an isomorphism.